

Cohomology and Deformation of Virasoro Extensions of q -Witt Hom-Lie superalgebra

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Abstract

The purpose of this paper is to study Virasoro extensions of the q -deformed Witt Hom-Lie superalgebra. Moreover, we provide the cohomology and deformations of the Ramond Hom-superalgebra and special Ramond Hom-superalgebra.

1 Introduction

Various important examples of Lie superalgebras have been constructed starting from the Witt algebra \mathcal{W} . It is well-known that \mathcal{W} (up to equivalence and rescaling) has a unique nontrivial one-dimensional central extension, the Virasoro algebra. This is not the case in the superalgebras case, very important examples are the Neveu-Schwarz and the Ramond superalgebras. For further generalizations, we refer to Schlichenmaier's book [9]. The Neveu-Schwarz and Ramond superalgebras are usually called super-Virasoro algebras since they can be viewed as super-analogs of the Virasoro algebra. Their corresponding second cohomology groups are computed in [4]. One may find the second cohomology group computation of Witt and Virasoro algebras in [6, 7, 8]. The q -deformed Witt superalgebra \mathcal{W}_q was defined in [1] as a main example of Hom-Lie superalgebras. The cohomology and deformations of \mathcal{W}_q were studied in [2, 3]. The first and second cohomology groups of the q -deformed Heisenberg-Virasoro algebra of Hom-type are computed in [5].

In this paper, we aim to study extensions of Hom-Lie superalgebras and discuss mainly the case of \mathcal{W}_q Hom-superalgebra. We provide a characterization of the Virasoro extensions of the q -Witt superalgebra and study their cohomology and deformations. In Section 2, we review the basics about Hom-Lie superalgebras and their cohomology; and in Section 3, we discuss their extensions. In Section 4, we describe the q -Witt superalgebra extensions of Virasoro type, we introduce Ramond Hom-superalgebra and special Ramond Hom-superalgebra. Section 5 is dedicated to cohomology and derivations calculations of Virasoro extensions of q -Witt superalgebra. In the last section we discuss one-parameter formal deformations of Ramond and special Ramond Hom-Lie superalgebras.

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2 Preliminaries

In this section, we recall definitions of Hom-Lie superalgebras, q -deformed Witt superalgebra and some basics about representations and cohomology. For more details we refer to [2].

Definition 2.1. A Hom-Lie superalgebra is a triple $(\mathcal{G}, [\cdot, \cdot], \alpha)$ consisting of a superspace \mathcal{G} , an even bilinear map $[\cdot, \cdot] : \mathcal{G} \times \mathcal{G} \rightarrow \mathcal{G}$ and an even superspace homomorphism $\alpha : \mathcal{G} \rightarrow \mathcal{G}$ satisfying

$$\begin{aligned} [x, y] &= -(-1)^{|x||y|}[y, x], \\ (-1)^{|x||z|}[\alpha(x), [y, z]] &+ (-1)^{|z||y|}[\alpha(z), [x, y]] + (-1)^{|y||x|}[\alpha(y), [z, x]] = 0, \end{aligned}$$

for all homogeneous element x, y, z in \mathcal{G} and where $|x|$ denotes the degree of the homogeneous element x .

2.1 A q -deformed Witt superalgebra

A q -deformed Witt superalgebra \mathcal{W}^q can be presented as the \mathbb{Z}_2 -graded vector space with $\{L_n\}_{n \in \mathbb{Z}}$ as a basis of the even homogeneous part and $\{G_n\}_{n \in \mathbb{Z}}$ as a basis of the odd homogeneous part. It is equipped with the commutator

$$[L_n, L_m] = (\{m\} - \{n\})L_{n+m}, \quad (2.1)$$

$$[L_n, G_m] = (\{m+1\} - \{n\})G_{n+m}, \quad (2.2)$$

where $\{m\}$ denotes the q -number m , that is $\{m\} = \frac{1-q^m}{1-q}$. The other brackets are obtained by supersymmetry or are equal to 0. The even linear map α on \mathcal{W}^q is defined on the generators by

$$\alpha(L_n) = (1 + q^n)L_n, \quad \alpha(G_n) = (1 + q^{n+1})G_n.$$

For more details, we refer to [1].

2.2 Cohomology of Hom-Lie superalgebras

Let $(\mathcal{G}, [\cdot, \cdot], \alpha)$ be a Hom-Lie superalgebra and $V = V_0 \oplus V_1$ be an arbitrary vector superspace. Let $\beta \in \mathcal{G}l(V)$ be an arbitrary even linear self-map on V and

$$\begin{aligned} [\cdot, \cdot]_V &: \mathcal{G} \times V \rightarrow V \\ (g, v) &\mapsto [g, v]_V \end{aligned}$$

a bilinear map satisfying $[G_i, V_j]_V \subset V_{i+j}$ where $i, j \in \mathbb{Z}_2$.

Definition 2.2. The triple $(V, [\cdot, \cdot]_V, \beta)$ is called a representation of the Hom-Lie superalgebra $\mathcal{G} = \mathcal{G}_0 \oplus \mathcal{G}_1$ or \mathcal{G} -module V if the even bilinear map $[\cdot, \cdot]_V$ satisfies, for $x, y \in \mathcal{G}$ and $v \in V$,

$$[[x, y], \beta(v)]_V = [\alpha(x), [y, v]]_V - (-1)^{|x||y|}[\alpha(y), [x, v]]_V. \quad (2.3)$$

Remark 2.3. When $[\cdot, \cdot]_V$ is the zero-map, we say that the module V is trivial.

Definition 2.4. [2] The set $C^k(\mathcal{G}, V)$ of k -cochains on space \mathcal{G} with values in V , is the set of k -linear maps $f : \otimes^k \mathcal{G} \rightarrow V$ satisfying

$$f(x_1, \dots, x_i, x_{i+1}, \dots, x_k) = -(-1)^{|x_i||x_{i+1}|} f(x_1, \dots, x_{i+1}, x_i, \dots, x_k) \text{ for } 1 \leq i \leq k-1.$$

For $k = 0$ we have $C^0(\mathcal{G}, V) = V$.

Define $\delta^k : C^k(\mathcal{G}, V) \rightarrow C^{k+1}(\mathcal{G}, V)$ by setting

$$\begin{aligned} \delta^k(f)(x_0, \dots, x_k) = & \sum_{0 \leq s < t \leq k} (-1)^{t+|x_t|(|x_{s+1}|+\dots+|x_{t-1}|)} f\left(\alpha(x_0), \dots, \alpha(x_{s-1}), [x_s, x_t], \alpha(x_{s+1}), \dots, \widehat{x_t}, \dots, \alpha(x_k)\right) \\ & + \sum_{s=0}^k (-1)^{s+|x_s|(|f|+|x_0|+\dots+|x_{s-1}|)} \left[\alpha^{k+r-1}(x_s), f\left(x_0, \dots, \widehat{x_s}, \dots, x_k\right) \right]_V, \end{aligned} \quad (2.4)$$

where $f \in C^k(\mathcal{G}, V)$, $|f|$ is the parity of f , $x_0, \dots, x_k \in \mathcal{G}$ and $\widehat{x_i}$ means that x_i is omitted.

We assume that the representation $(V, [\cdot, \cdot]_V, \beta)$ of a Hom-Lie superalgebra $(\mathcal{G}, [\cdot, \cdot], \alpha)$ is trivial. Since $[\cdot, \cdot]_V = 0$, the operator defined in (2.4) becomes

$$\delta_T^k(f)(x_0, \dots, x_k) = \sum_{0 \leq s < t \leq k} (-1)^{t+|x_t|(|x_{s+1}|+\dots+|x_{t-1}|)} f\left(\alpha(x_0), \dots, \alpha(x_{s-1}), [x_s, x_t], \alpha(x_{s+1}), \dots, \widehat{x_t}, \dots, \alpha(x_k)\right). \quad (2.5)$$

The pair $(\oplus_{k \geq 0} C_{\alpha, \beta}^k(\mathcal{G}, V), \{\delta^k\}_{k \geq 0})$ defines a chomology complex, that is $\delta^k \circ \delta^{k-1} = 0$.

- The k -cocycles space is defined as $Z^k(\mathcal{G}) = \ker \delta^k$.
- The k -coboundary space is defined as $B^k(\mathcal{G}) = \text{Im } \delta^{k-1}$.
- The k^{th} cohomology space is the quotient $H^k(\mathcal{G}) = Z^k(\mathcal{G})/B^k(\mathcal{G})$. It decomposes as well as even and odd k^{th} cohomology spaces.

Now, we consider the adjoint representation of a Hom-superalgebra and define the first and second coboundary maps. For all $f \in C_{\alpha, \alpha}^1(\mathcal{G}, \mathcal{G}) = \{g \in C^1(\mathcal{G}, \mathcal{G}); g \circ \alpha = \alpha \circ g\}$ the operator defined in (2.4) ($r = 0$, $k \in \{1, 2\}$) becomes

$$\delta_{\mathcal{G}}^1(f)(x, y) = -f([x, y]) + (-1)^{|x||f|}[x, f(y)] - (-1)^{|y|(|f|+|x|)}[y, f(x)] \quad (2.6)$$

$$\begin{aligned} \delta_{\mathcal{G}}^2(f)(x, y, z) = & -f([x, y], \alpha(z)) + (-1)^{|z||y|} f([x, z], \alpha(y)) + f(\alpha(x), [y, z]) \\ & + (-1)^{|x||f|}[\alpha(x), f(y, z)] - (-1)^{|y|(|f|+|x|)}[\alpha(y), f(x, z)] \\ & + (-1)^{|z|(|f|+|x|+|y|)}[\alpha(z), f(x, y)] \end{aligned} \quad (2.7)$$

Then we have

$$\delta_{\mathcal{G}}^2 \circ \delta_{\mathcal{G}}^1(f) = 0, \quad \forall f \in C_{\alpha, \alpha}^1(\mathcal{G}, \mathcal{G}).$$

We denote by $H^1(\mathcal{G}, \mathcal{G})$ (resp. $H^2(\mathcal{G}, \mathcal{G})$) the corresponding 1st and 2nd cohomology groups. An element f of $Z^1(\mathcal{G}, \mathcal{G})$ is called derivation of \mathcal{G} .

3 Extensions of Hom-Lie superalgebras

An extension of a Hom-Lie superalgebra $(\mathcal{G}, [\cdot, \cdot], \alpha)$ by a representation $(V, [\cdot, \cdot]_V, \beta)$ is an exact sequence

$$0 \longrightarrow (V, \beta) \xrightarrow{i} (K, \gamma) \xrightarrow{\pi} (\mathcal{G}, \alpha) \longrightarrow 0$$

satisfying $\gamma \circ i = i \circ \beta$ and $\alpha \circ \pi = \pi \circ \gamma$.

This extension is said to be central if $[K, i(V)]_K = 0$.

In particular, if $K = \mathcal{G} \oplus V$, $i(v) = v$, $\forall v \in V$ and $\pi(x) = x$, $\forall x \in \mathcal{G}$, then we have $\gamma(x, v) = (\alpha(x), \beta(v))$ and we denote

$$0 \longrightarrow (V, \beta) \longrightarrow (K, \gamma) \longrightarrow (\mathcal{G}, \alpha) \longrightarrow 0.$$

For convenience, we introduce the following notation for certain cochains spaces on $K = \mathcal{G} \oplus V$, $\mathcal{C}(\mathcal{G}^n, \mathcal{G})$ and $\mathcal{C}(\mathcal{G}^k V^l, V)$, where $\mathcal{G}^k V^l$ is the subspace of K^{k+l} determined by products of k elements from \mathcal{G} and l elements from V .

Let $(\phi, \psi) \in \mathcal{C}^2(K, K) \times \mathcal{C}^2(K, K)$. We set, for all $X, Y, Z \in K_0 \cup K_1$,

$$\phi \circ \psi(X, Y, Z) = (-1)^{|X||Z|} \odot_{X,Y,Z} (-1)^{|X|(|\psi|+|Z|)} \phi(\gamma(X), \psi(Y, Z)),$$

and

$$[\phi, \psi] = \phi \circ \psi + (-1)^{|\psi||\phi|} \psi \circ \phi.$$

For $f \in \mathcal{C}^2(K, K)$, we set $f = \tilde{f} + \hat{f} + \bar{f} + v + \hat{v} + \bar{v}$ where $\tilde{f} \in \mathcal{C}(\mathcal{G}^2, \mathcal{G})$, $\hat{f} \in \mathcal{C}(\mathcal{G}V, \mathcal{G})$, $\bar{f} \in \mathcal{C}(V^2, \mathcal{G})$, $v \in \mathcal{C}(\mathcal{G}^2, V)$, $\hat{v} \in \mathcal{C}(\mathcal{G}V, V)$ and $\bar{v} \in \mathcal{C}(V^2, V)$.

Let $d \in \mathcal{C}^2(K, K)$, if (K, d, γ) is a Hom-Lie superalgebra and V is an ideal in K (i.e. $d(\mathcal{G}, V) \subset V$), we obtain by using the above notation :

- $\hat{d} \equiv 0$,
- $\bar{d} \equiv 0$,
- $0 = [d, d](x, y, z) = \left([\tilde{d}, \tilde{d}] + 2[v, \tilde{d}] + 2[\hat{v}, v] \right)(x, y, z), \quad \forall x, y, z \in \mathcal{G},$
 $[\tilde{d}, \tilde{d}](x, y, z) \in \mathcal{G}$ and $\left(2[v, \tilde{d}] + 2[\hat{v}, v] \right)(x, y, z) \in V,$
- $0 = [d, d](x, y, w) = \left([\hat{v}, \hat{v}] + 2[\tilde{d}, \hat{v}] + 2[\bar{v}, v] \right)(x, y, w), \quad \forall (x, y, w) \in \mathcal{G}^2 \times V,$
- $0 = [d, d](x, v, w) = 2[\bar{v}, \hat{v}](x, v, w), \quad \forall (x, v, w) \in \mathcal{G} \times V^2,$
- $0 = [d, d](u, v, w) = \frac{1}{2}[\bar{v}, \bar{v}](u, v, w), \quad \forall (u, v, w) \in V^3.$

We deduce the following theorem

Theorem 3.1. *The triple (K, d, γ) is a Hom-Lie superalgebra if and only if the following conditions are satisfied*

- $(\mathcal{G}, \tilde{d}, \alpha)$ is a Hom-Lie superalgebra,
- $[v, \tilde{d}] + [\hat{v}, v] \equiv 0$,

- $\frac{1}{2}[\widehat{v}, \widehat{v}] + [\widetilde{d}, \widehat{v}] + [\overline{v}, v] \equiv 0,$
- $[\overline{v}, \widehat{v}] \equiv 0,$
- (V, \overline{v}, β) is a Hom-Lie superalgebra.

Corollary 3.2. *If $\overline{v} \equiv 0$, then the triple (K, d, γ) is a Hom-Lie superalgebra if and only if the following conditions are satisfied*

- $(\mathcal{G}, \widetilde{d}, \alpha)$ is a Hom-Lie superalgebra,
- (V, \widehat{v}, β) is a representation of \mathcal{G} ,
- v is a 2-cocycle on V (with the cohomology defined by $(\mathcal{G}, \widetilde{d}, \alpha)$ and (V, \widehat{v}, β)).

Corollary 3.3. *Let*

$$0 \longrightarrow (V, \beta) \longrightarrow (\mathcal{G} \oplus V, \widetilde{\alpha}) \longrightarrow (\mathcal{G}, \alpha) \longrightarrow 0$$

be an extension of $(\mathcal{G}, [\cdot, \cdot], \alpha)$ by a representation $(V, [\cdot, \cdot]_V, \beta)$, where $\widetilde{\alpha}$ is defined by $\widetilde{\alpha}(x, v) = (\alpha(x), \beta(v))$, for all $x \in \mathcal{G}$ and $v \in V$.

Let $\varphi \in (C^2(\mathcal{G}, V))_j$, ($j \in \mathbb{Z}_2$). We define a skew-symmetric bilinear bracket operation $d : \wedge^2(\mathcal{G} \oplus V) \rightarrow \mathcal{G} \oplus V$ by

$$d((x, u); (y, v)) = ([x, y], [x, v]_V - (-1)^{|u||y|}[y, u]_V + \varphi(x, y)) \quad \forall x, y \in \mathcal{G} \quad v, w \in V. \quad (3.1)$$

The triple $(\mathcal{G} \oplus V, d, \widetilde{\alpha})$ is a Hom-Lie superalgebra if and only if φ is a 2-cocycle (i.e. $\varphi \in Z^2(\mathcal{G}, V)$).

Proof. We have $\overline{v} \equiv 0$, $\widetilde{d} = [\cdot; \cdot]$, $v = \varphi$, $\widehat{v} = [\cdot, \cdot]_V$ and $\overline{v} \equiv 0$. Then, we deduce

- $(\mathcal{G}, \widetilde{d}, \alpha)$ is a Hom-Lie superalgebra
- (V, \widehat{v}, β) is a representation of \mathcal{G} .

Therefore, the triple $(\mathcal{G} \oplus V, d, \gamma)$ is a Hom-Lie superalgebra if and only if φ is a 2-cocycle (i.e. $\varphi \in Z^2(\mathcal{G}, V)$). ■

Remark 3.4. • If φ is even, then $\mathcal{G}_0 \oplus V_0$ is an even homogeneous part and $\mathcal{G}_1 \oplus V_1$ is the odd homogeneous part of $\mathcal{G} \oplus V$.

- If φ is odd, then, $\mathcal{G}_0 \oplus V_1$ is an even homogeneous part and $\mathcal{G}_1 \oplus V_0$ is the odd homogeneous part of $\mathcal{G} \oplus V$. The Hom-Lie superalgebra $\mathcal{G} \oplus V$ is called the special extension of \mathcal{G} by V .

Theorem 3.5. *Let $(V, [\cdot, \cdot]_V, \beta)$ be a representation of a Hom-Lie superalgebra $(\mathcal{G}, [\cdot, \cdot], \alpha)$. The even second cohomology space $H_0^2(\mathcal{G}, V) = Z_0^2(\mathcal{G}, V)/B_0^2(\mathcal{G}, V)$ is in one-to-one correspondence with the set of the equivalence classes extensions of $(\mathcal{G}, [\cdot, \cdot], \alpha)$ by $(V, [\cdot, \cdot]_V, \beta)$.*

Proof. Let $(\mathcal{G} \oplus V, d, \gamma)$ and $(\mathcal{G} \oplus V, d', \gamma)$ be two extensions of $(\mathcal{G}, [\cdot, \cdot], \alpha)$. So there are two even cocycles φ and φ' such as $d((x, u); (y, v)) = ([x, y], [x, v]_V + [u, y]_V + \varphi(x, y))$ and $d'((x, u); (y, v)) = ([x, y], [x, v]_V + [u, y]_V + \varphi'(x, y))$. If $\varphi - \varphi' = \delta^1 h(x, y)$, where $h : \mathcal{G} \rightarrow V$ is a linear map satisfying $h \circ \alpha = \beta \circ h$ (i.e. $\varphi - \varphi' \in B^2(\mathcal{G}, V)$). Let us define $\Phi : (\mathcal{G} \oplus V, d, \gamma) \rightarrow (\mathcal{G} \oplus V, d', \gamma)$ by $\Phi(x, v) = (x, v - h(x))$. It is clear that Φ is bijective. Let us check that Φ is a Hom-Lie superalgebras homomorphism. We have

$$\begin{aligned}
& d(\Phi((x, v)), \Phi((y, w))) \\
&= d((x, v - h(x)), (y, w - h(y))) \\
&= ([x, y], [x, w]_V + [v, y]_V - [x, h(y)]_V - [h(x), y]_V + \varphi(x, y)) \\
&= ([x, y], [x, w]_V + [v, y]_V - \delta^1(h)(x, y) + \varphi(x, y) - h([x, y])) \\
&= \Phi([x, y], f(x, y)) \\
&= ([x, y], [x, w]_V + [v, y]_V + \varphi'(x, y) - h([x, y])) \\
&= \Phi([x, y], [x, w]_V + [v, y]_V + \varphi'(x, y)) \\
&= \Phi(d'((x, v), (y, w))).
\end{aligned}$$

■

Definition 3.6. • If $\varphi \equiv 0$, the Hom-Lie superalgebra $(\mathcal{G} \oplus V, d, \gamma)$ is called the semidirect product of the Hom-Lie superalgebra $(\mathcal{G}, [\cdot, \cdot], \alpha)$ and $(V, [\cdot, \cdot]_V, \beta)$.

- If $V = c\mathbb{C}$ ($c \in \mathbb{C}$) and $[\cdot, \cdot]_V \equiv 0$, the Hom-Lie superalgebra $(\mathcal{G} \oplus V, d, \gamma)$ is called a one-dimensional central extension of \mathcal{G} .
- If $V = c\mathbb{C}$, $[\cdot, \cdot]_V \equiv 0$ and $\mathcal{G} = \mathcal{W}^q$, the Hom-Lie superalgebra $(\mathcal{G} \oplus V, d, \gamma)$ is called a Virasoro Hom-superalgebra.
- If $V = c\mathbb{C}$, $[\cdot, \cdot]_V \equiv 0$, $\varphi \equiv 0$ and $\mathcal{G} = \mathcal{W}^q$, the Hom-Lie superalgebra $(\mathcal{G} \oplus V, d, \gamma)$ is called a trivial Virasoro Hom-superalgebra.

Definition 3.7. A Hom-Lie superalgebra $(\mathcal{G}, [\cdot, \cdot], \alpha)$ is said to be \mathbb{Z} -graded if $\mathcal{G} = \bigoplus_{n \in \mathbb{Z}} \mathcal{G}_n$, where $\dim(\mathcal{G}_n) < \infty$, $\alpha(\mathcal{G}_n) \subset \mathcal{G}_n$ and $[\mathcal{G}_n, \mathcal{G}_m] \subset \mathcal{G}_{n+m}$, for all $n, m \in \mathbb{Z}$. For an element $x \in \mathcal{G}$, we call n the degree of x , denoted $\deg(x) = n$, if $x \in \mathcal{G}_n$.

Proposition 3.8. Let $(\mathcal{G}, [\cdot, \cdot], \alpha)$ be a \mathbb{Z} -graded Hom-Lie superalgebra. We denote $((\mathcal{G} \oplus c\mathbb{C})_0 = \mathcal{G}_0 \oplus \mathbb{C}$ and $(\mathcal{G} \oplus c\mathbb{C})_n = \mathcal{G}_n$, $\forall n \in \mathbb{Z}^*$. If $\varphi(\mathcal{G}_n, \mathcal{G}_m) \subset \delta_{n+m, 0} \mathcal{G}_{n+m}$ then $(\mathcal{G} \oplus c\mathbb{C}, d, \gamma)$ is also \mathbb{Z} -graded.

4 Virasoro Hom-superalgebras

In this section, we describe extensions of q -deformed Witt superalgebra \mathcal{W}^q . Let us recall the following result describing its scalar cohomology:

Theorem 4.1. [2]

$$H^2(\mathcal{W}^q, \mathbb{C}) = \mathbb{C}[\varphi_0] \oplus \mathbb{C}[\varphi_1],$$

where

$$\begin{aligned}\varphi_0(xL_n + yG_m, zL_p + tG_k) &= xzb_n\delta_{n+p,0}, \\ \varphi_1(xL_n + yG_m, zL_p + tG_k) &= xtb_n\delta_{n+k,-1} - yzb_p\delta_{p+m,-1},\end{aligned}$$

with

$$b_n = \begin{cases} \frac{1}{q^{n-2}} \frac{1+q^2}{1+q^n} \frac{\{n+1\}\{n\}\{n-1\}}{\{3\}\{2\}}, & \text{if } n \geq 0, \\ -b_{-n} & \text{if } n < 0. \end{cases}$$

Using the even non trivial 2-cocycle φ_0 defined in Theorem 4.1, we can define the followings Virasoro Hom-superalgebras :

- The *Neveu-Schwarz Hom-superalgebra* can be presented as the \mathbb{Z}_2 -graded vector space with $\{L_n, D\}_{n \in \mathbb{Z}}$ as a basis of the even homogeneous part and $\{F_n\}_{n \in \frac{1}{2} + \mathbb{Z}}$ as a basis of the odd homogeneous part. It is equipped with the commutator

$$\begin{aligned}[L_n, L_m] &= (\{m\} - \{n\})L_{n+m} + D \frac{1}{q^{n-2}} \frac{1+q^2}{1+q^n} \frac{\{n+1\}\{n\}\{n-1\}}{\{12\}} \delta_{n+m,0}, \\ &\quad \forall (n, m) \in (\mathbb{Z}_+ \times \mathbb{Z}) \cup (\mathbb{Z}_- \times \mathbb{Z}_-) \\ [L_n, F_m] &= (\{m + \frac{1}{2}\} - \{n\})F_{n+m} \\ [F_n, F_m] &= [F_n, D] = [L_n, D] = 0.\end{aligned}$$

- The *Ramond Hom-superalgebra* satisfies the following commutation relations:

$$\begin{aligned}[L_n + z, L_m + z'] &= (\{m\} - \{n\})L_{n+m} + c \frac{1}{q^{n-2}} \frac{1+q^2}{1+q^n} \frac{\{n+1\}\{n\}\{n-1\}}{\{12\}} \delta_{n+m,0}, \forall n > 0 \\ [L_n + z, G_m + z'] &= (\{m+1\} - \{n\})G_{n+m} \\ [G_n + z, G_m + z'] &= 0.\end{aligned}$$

Using the odd non trivial 2-cocycle φ_1 defined in Theorem 4.1, we can define the *special Ramond Hom-superalgebra*. Then it is equipped with the commutator

$$\begin{aligned}[L_n + z, L_m + z'] &= (\{m\} - \{n\})L_{n+m} \\ [L_n + z, G_m + z'] &= (\{m+1\} - \{n\})G_{n+m} + c \frac{1}{q^{n-2}} \frac{1+q^2}{1+q^n} \frac{\{n+1\}\{n\}\{n-1\}}{\{12\}} \delta_{n+m,0}, \forall n > 0 \\ [G_n + z, G_m + z'] &= 0.\end{aligned}$$

In the above cases, the map γ is defined by

$$\gamma(x, z) = (\alpha(x), z), \quad \forall x \in \mathcal{W}^q, \forall z \in \mathbb{C},$$

where α is given in Section 2.1.

Remark 4.2. We denote the Ramond Hom-Lie superalgebra by HR , the Neveu-Schwarz Hom-Lie superalgebra by HN and the special Ramond Hom-superalgebra by SHR . The map $f : HR \rightarrow HN$ defined by $f(L_n) = L_n$, $f(G_n) = F_{n+\frac{1}{2}}$ and $f(c) = D$ is a Hom-Lie superalgebras isomorphism.

Hence, Virasoro Hom-superalgebras are characterized in the following Theorem.

Theorem 4.3. *Every Virasoro Hom-superalgebra is isomorphic to one of the following Virasoro Hom-Lie superalgebras :*

- *Ramond Hom-superalgebra,*
- *Special Ramond Hom-superalgebra,*
- *Trivial Virasoro Hom-superalgebra.*

5 Cohomology of Extensions of Hom-Lie superalgebras and Virasoro Hom-superalgebras

Let

$$0 \longrightarrow (V, \beta) \longrightarrow (K, d, \gamma) \longrightarrow (\mathcal{G}, \alpha) \longrightarrow 0$$

be an extension of $(\mathcal{G}, \delta, \alpha)$ by a representation (V, λ, β) , where $K = \mathcal{G} \oplus V$ and $d = \delta + \lambda + \varphi$ ($\varphi \in Z^2(\mathcal{G}, V)$).

5.1 Derivations of Extensions of Hom-Lie superalgebras

If $f \in \mathcal{C}^1(K, K)$, we set $f = \tilde{f} + \hat{f} + v + \hat{v}$ where $\tilde{f} \in \mathcal{C}^1(\mathcal{G}, \mathcal{G})$, $\hat{f} \in \mathcal{C}^1(V, \mathcal{G})$, $v \in \mathcal{C}^1(\mathcal{G}, V)$ and $\hat{v} \in \mathcal{C}^1(V, V)$. If $(\phi, \psi) \in \mathcal{C}^2(K, K) \times \mathcal{C}^1(K, K)$, we define

$$\phi \circ \psi(X, Y) = \phi(\psi(X), \gamma^r(Y),) - (-1)^{|X||Y|} \phi(\psi(Y), \gamma^r(X),) \quad \forall X, Y \in K_0 \cup K_1$$

and

$$[\phi, \psi] = \phi \circ \psi - (-1)^{|\psi||\phi|} \psi \circ \phi.$$

For $f \in \mathcal{C}^1(K, K)$, we have

$$\begin{aligned} \delta_K^1(f)((x+u, y+v)) = \\ -f(d(x+u, y+v)) + d(f(x+u), \gamma^r(y+v)) + (-1)^{|f||x+u|} d(\gamma^r(x+u), f(y+v)), \end{aligned}$$

which implies $[d, f] = \delta_K^1(f)$.

Then

$$(f \text{ is a } \alpha^r\text{-derivation of } K) \Leftrightarrow ([d, f] \equiv 0). \quad (5.1)$$

For all $x, y \in \mathcal{G}$, $u, v \in V$, we have

$$\begin{aligned} [d, f](x, y) &= ([\delta, \tilde{f}] + [\delta + \lambda, v] + [\varphi, \hat{f} + \hat{v}])(x, y), \\ [d, f](x, v) &= ([\delta, \hat{f}] + [\lambda, \tilde{f} + \hat{f} + \hat{v}] + [\varphi, \hat{f}])(x, v), \\ [d, f](u, v) &= 0. \end{aligned}$$

Then, we deduce the following result :

Theorem 5.1. For all $x, y \in \mathcal{G}$, $v \in V$, we have

$$(f \text{ is a } \alpha^r\text{-derivation of } K) \Leftrightarrow \begin{cases} ([\delta, \tilde{f}] + [\varphi, \widehat{f}])(x, y) = 0, \\ ([\varphi, \tilde{f} + \widehat{v}] + [\delta + \lambda, v])(x, y) = 0, \\ [\delta + \lambda, \widehat{f}](x, v) = 0, \\ ([\lambda, \tilde{f} + \widehat{v}] + [\varphi, \widehat{f}])(x, v) = 0. \end{cases} \quad (5.2)$$

5.2 Derivations of Virasoro Hom-superalgebras

Let recall the following result (for the proof see [2]).

Lemma 5.2. The set of α^0 -derivations of the Hom-Lie superalgebra \mathcal{W}^q is

$$Der_{\alpha^0}(\mathcal{W}^q) = \langle D_1 \rangle \oplus \langle D_2 \rangle \oplus \langle D_3 \rangle \oplus \langle D_4 \rangle$$

where D_1, D_1, D_2, D_3 and D_4 are defined, with respect to the basis as

$$\begin{aligned} D_1(L_n) &= nL_n, & D_1(G_n) &= nG_n, \\ D_2(L_n) &= 0, & D_2(G_n) &= G_n, \\ D_3(L_n) &= nG_{n-1}, & D_3(G_n) &= 0, \\ D_4(L_n) &= 0, & D_4(G_n) &= L_{n+1}. \end{aligned}$$

Let $(\mathcal{W}_\varphi^q, d, \gamma)$ be a Hom-Virasoro-superalgebra. Then

$$\mathcal{W}_\varphi^q = \mathcal{W}^q \oplus \mathbb{C}, \quad d = [\cdot, \cdot] + \varphi, \quad \text{and } \gamma(x, z) = (\alpha(x), z),$$

where $\varphi \in \mathbb{C}\varphi_0 \cup \mathbb{C}\varphi_1$.

Lemma 5.3.

$$(f \text{ is a } \alpha^r\text{-derivation of } \mathcal{W}_\varphi^q) \Rightarrow (\widehat{f} \equiv 0).$$

Proof. Let f be an α^r -derivation of \mathcal{W}_φ^q .

Using Theorem 5.1 and $\lambda \equiv 0, \forall z \in \mathbb{C}, \forall n \in \mathbb{Z}$ we have

$$\begin{aligned} [\delta, \widehat{f}](L_n, z) &= 0 \\ \Rightarrow -(-1)^{|x||z|} \delta(\widehat{f}(z), \alpha(L_n)) &= 0, \\ \Rightarrow [\widehat{f}(z), L_n] &= 0, \\ \Rightarrow \widehat{f}(z) &= 0. \end{aligned}$$

■

In the following, we provide the α^0 -derivations of \mathcal{W}_φ^q explicitly.

Proposition 5.4. The set of α^0 -derivations of the Virasoro Hom-Lie superalgebra $\mathcal{W}_{\varphi_i}^q$ with $i = 0, 1$, is

$$Der_{\alpha^0}(\mathcal{W}_{\varphi_i}^q) = \langle D_1 \rangle \oplus \langle D_2 \rangle \oplus \langle D_3 \rangle \oplus \langle D_4 \rangle,$$

where D_1, D_1, D_2, D_3 and D_4 are defined, with respect to the basis, as

$$\begin{aligned} D_1(L_n) &= nL_n, & D_1(G_n) &= nG_n, & D_1(1) &= 0, \\ D_2(L_n) &= 0, & D_2(G_n) &= G_n, & D_2(1) &= 0, \\ D_3(L_n) &= nG_{n-1}, & D_3(G_n) &= 0, & D_3(1) &= 0, \\ D_4(L_n) &= 0, & D_4(G_n) &= L_{n+1}, & D_4(1) &= 0. \end{aligned}$$

Proof. Using Lemma 5.2 and the first equation in (5.2), we obtain

$$\tilde{f} \in \langle D_1 \rangle \oplus \langle D_2 \rangle \oplus \langle D_3 \rangle \oplus \langle D_4 \rangle.$$

If \tilde{f} is even, there exist λ_1 and λ_2 satisfying $\tilde{f} = \lambda_1 D_1 + \lambda_2 D_2$.

If \tilde{f} is odd, there exist λ_3 and λ_4 satisfying $\tilde{f} = \lambda_3 D_3 + \lambda_4 D_4$.

Using (5.2), $\lambda \equiv 0$ and $\hat{f} \equiv 0$, we obtain

$$\begin{aligned} & \left([\varphi, \tilde{f} + \hat{v}] + [\delta, v] \right)(x, y) = 0 \\ \Rightarrow & \varphi(\tilde{f}(x), \alpha(y)) - (-1)^{|x||y|} \varphi(\tilde{f}(y), \alpha(x)) - \hat{v}(\varphi(x, y)) - v(\delta(x, y)) = 0. \\ \Rightarrow & \begin{cases} \varphi(\tilde{f}(L_n), \alpha(L_k)) - \varphi(\tilde{f}(L_k), \alpha(L_n)) - \hat{v}(\varphi(L_n, L_k)) - v(\delta(L_n, L_k)) = 0, \\ \varphi(\tilde{f}(L_k), \alpha(G_n)) - \varphi(\tilde{f}(G_n), \alpha(L_k)) - \hat{v}(\varphi(L_k, G_n)) - v(\delta(L_k, G_n)) = 0. \end{cases} \end{aligned}$$

With $\varphi \in \{\varphi_0, \varphi_1\}$, we have :

$$\begin{aligned} \varphi(L_1, x) &= 0, & \forall x \in \mathcal{W}^q, \\ \varphi(L_n, L_k) &= 0, & \forall n + k \neq 0, \\ \varphi(L_n, G_k) &= 0, & \forall n + k \neq -1. \end{aligned}$$

Since $\tilde{f} \in \{\lambda_1 D_1 + \lambda_2 D_2, \lambda_3 D_3 + \lambda_4 D_4\}$, we obtain

$$\begin{cases} v(\delta(L_n, L_k)) = 0, \\ v(\delta(L_k, G_n)) = 0. \end{cases}$$

Thus, we can deduce $v \equiv 0$ and $\hat{v} \equiv 0$ ■

5.3 2-cocycles of Extensions of Hom-Lie superalgebras

Let $f \in \mathcal{C}^2(K, K)$, we set $f = \tilde{f} + \hat{f} + \overline{f} + v + \hat{v} + \overline{v}$ where $\tilde{f} \in \mathcal{C}^2(\mathcal{G}, \mathcal{G})$, $\hat{f} \in \mathcal{C}^{1,1}(\mathcal{G}V, \mathcal{G})$, $\overline{f} \in \mathcal{C}^2(V, \mathcal{G})$, $v \in \mathcal{C}^2(\mathcal{G}, V)$, $\hat{v} \in \mathcal{C}^{1,1}(\mathcal{G}V, V)$ and $\overline{v} \in \mathcal{C}^2(V, V)$. In this case, we have $[d, f] = \delta_K$. For all $x, y, z \in \mathcal{G}$, $u, v, w \in V$, we have

$$\begin{aligned} [d, f](x, y, z) &= \left([\delta, \tilde{f}] + [\varphi, \hat{f}] + [\delta + \lambda, v] + [\varphi, \tilde{f} + \hat{v}] \right)(x, y, z) \\ & \left([\delta, \tilde{f}] + [\varphi, \hat{f}] \right)(x, y, z) \in \mathcal{G} \\ & \left([\delta + \lambda, v] + [\varphi, \tilde{f} + \hat{v}] \right)(x, y, z) \in V \\ [d, f](x, y, w) &= \left([\delta, \hat{f} + \hat{v}] + [\varphi, \overline{f}] + [\lambda, \tilde{f} + \hat{f} + \hat{v}] + [\varphi, \hat{f} + \hat{v} + \overline{v}] \right)(x, y, w) \\ & \left([\delta, \overline{f}] + [\lambda, \hat{f} + \overline{f} + \overline{v}] + [\varphi, \overline{f} + \overline{v}] \right)(x, u, v) = 0 \\ [d, f](u, v, w) &= 0. \end{aligned}$$

Therefore, we have the following result

Theorem 5.5. For all $x, y, z \in \mathcal{G}$, $u, v, w \in V$, we have

$$f \in Z^2(K, K) \Leftrightarrow \begin{cases} [\delta, \tilde{f}] + [\varphi, \hat{f}](x, y, z) = 0, \\ [\delta + \lambda, v] + [\varphi, \tilde{f} + \hat{v}](x, y, z) = 0, \\ ([\delta + \lambda, \hat{f}] + [\varphi, \bar{f}])(x, y, v) = 0, \\ [\delta + \lambda, \hat{v}] + [\lambda, \tilde{f}] + [\varphi, \hat{f} + \bar{v}](x, y, v) = 0, \\ [\delta, \bar{f}](x, u, v) = 0, \\ ([\lambda, \hat{f} + \bar{f} + \bar{v}] + [\varphi, \bar{f} + \bar{v}])(x, u, v) = 0, \\ [\lambda, \bar{f}](u, v, w) = 0. \end{cases} \quad (5.3)$$

Corollary 5.6. 1. If $f = \tilde{f}$, then

$$f \in Z^2(K, K) \Leftrightarrow \begin{cases} [\delta, \tilde{f}](x, y, z) = 0, \\ [\varphi, \tilde{f}](x, y, z) = 0, \\ [\lambda, \tilde{f}](x, y, v) = 0. \end{cases} \quad (5.4)$$

2. If $f = v$, then

$$(f \in Z^2(K, K)) \Leftrightarrow (f \in Z^2(\mathcal{G}, V)) \quad (5.5)$$

Now, we assume that \mathcal{G} is \mathbb{Z} -graded and $\deg(\varphi) = 0$. Let $s = \deg(\tilde{f}) = \deg(\hat{f})$. If $x \in \mathcal{G}_n$, we set $x_n = x$.

In the first equation in (5.3), we have :

- $\deg([\delta, \tilde{f}](x_n, x_m, x_p)) = n + m + p + s,$
- $\deg(\hat{f}(\alpha(x_m), \varphi(x_p, x_n))) = m + s,$
- $\deg(\hat{f}(\alpha(x_n), \varphi(x_m, x_p))) = n + s,$
- $\deg(\hat{f}(\alpha(x_p), \varphi(x_n, x_m))) = p + s.$

In the third equation in (5.3), we have :

- $\deg([\delta, \hat{f}]) = n + m + s,$
- $\deg(\hat{f}(\alpha(x_m), \lambda(x_n, v))) = n + s,$
- $\deg(\hat{f}(\alpha(x_m), \lambda(x_n, v))) = m + s,$
- $\deg(\bar{f}(\beta(v), \varphi(x_n, x_m))) = s.$

The other terms are of degree zero. Then, we get

Theorem 5.7. If $f \in Z^2(K, K)$ and $n \neq p, n \neq m, p \neq m$, we have :

$$[\delta, \tilde{f}](x_n, x_m, x_p) = 0; \forall n + m \neq 0, n + p \neq 0, m + p \neq 0, \quad (5.6)$$

$$[\delta, \tilde{f}](x_n, x_{-n}, x_p) + \hat{f}(\alpha(x_p), \varphi(x_n, x_{-n})) = 0, \quad (5.7)$$

$$\hat{f}(\alpha(x_n), \varphi(x_m, x_p)) = 0, m + p \neq 0, m \neq 0, \quad (5.8)$$

$$[\delta, \hat{f}](x_n, x_m, u) = 0 \quad \forall n + m \neq 0, n \neq 0, m \neq 0, \quad (5.9)$$

$$\hat{f}(\alpha(x_n), \lambda(x_m, u)) = 0 \quad \forall n \neq m, n \neq 0, m \neq 0, \quad (5.10)$$

$$[\delta, \hat{f}](x_0, x_m, u) + \hat{f}(\alpha(x_m), \lambda(u, x_0)) = 0, \forall m \neq 0, \quad (5.11)$$

$$\bar{f}(\beta(u), \varphi(x_n, x_m)) = 0, \forall n + m \neq 0, n \neq 0, m \neq 0, \quad (5.12)$$

$$\delta(\alpha(x_n), \bar{f}(u, v)) = 0, \forall n \neq 0. \quad (5.13)$$

5.4 Second cohomology of Ramond Hom-superalgebra

Let f be an even 2-cocycle of degree s . We can assume that

$$\begin{aligned}\tilde{f}(L_n, L_p) &= a_{s,n,p}L_{s+n+p}, \quad \tilde{f}(L_n, G_p) = b_{s,n,p}G_{s+n+p}, \quad \tilde{f}(G_n, G_p) = c_{s,n,p}L_{s+n+p}, \\ \hat{f}(1, L_p) &= a'_{s,p}L_{s+p} \text{ and } \hat{f}(1, G_p) = b'_{s,p}G_{s+p}.\end{aligned}$$

Using (5.9), $\lambda \equiv 0$ and (5.11), we obtain the following equation

$$(\{m\} - \{n\})a'_{s,n+m} = (\{m\} - \{n+s\})a'_{s,n} + (\{m+s\} - \{n\})a'_{s,m}. \quad (5.14)$$

This equation was solved in [2]. The solutions are given by $a'_{s,n} = 0$ if $s \neq 0$ and $a'_{0,n} = na'_{0,1}$ if $s = 0$.

As above, we obtain $b'_{s,n} = 0$ if $s \neq 0$. $b'_{0,n} = b'_{0,0} + na'_{0,1}$.

Lemma 5.8. *If f is a 2-cocycle, we have*

$$\bar{v} \equiv 0, \quad \hat{f} \equiv 0, \quad \bar{f} \equiv 0, \quad \text{and} \quad \hat{v} \equiv 0.$$

Proof. Since \mathbb{C} is continued in even part of HR , $\forall z, z' \in \mathbb{C}$; we have

$$\bar{f}(z, z') = zz' \bar{f}(1, 1) = 0 \quad \text{and} \quad \bar{v}(z, z') = zz' \bar{v}(1, 1) = 0.$$

Let $g = \hat{f}(1, \cdot)$. If we assume $[d, f](x_n, x_m, 1) = 0$, we can deduce

$$\begin{aligned}-[\alpha(x_n), g(x_m)] + [\alpha(x_m), g(x_n)] + g([x_n, x_m]) - \varphi(\alpha(x_n), g(x_m)) \\ + \varphi(\alpha(x_m), g(x_n)) + \hat{v}((1, [x_n, x_m])) = 0.\end{aligned}$$

Since $[\alpha(x_n), g(x_m)] + [\alpha(x_m), g(x_n)] + g([x_n, x_m]) \in \mathcal{W}^q$ and $-\varphi(\alpha(x_n), g(x_m)) + \varphi(\alpha(x_m), g(x_n)) + \hat{v}((1, [x_n, x_m])) \in \mathbb{C}$, we obtain

$$-[\alpha(x_n), g(x_m)] + [\alpha(x_m), g(x_n)] + g([x_n, x_m]) = 0.$$

Then g is an α -derivation.

Recall that the set of α -derivations is trivial (see[2]). Therefore $g \equiv 0$ and $\hat{v}(1, \cdot) \equiv 0$. ■

Theorem 5.9. $H^2(HR, HR) = \mathbb{C}[\varphi_1]$.

Proof. We have $H^2(\mathcal{W}^q, \mathcal{W}^q) = \{0\}$ (see [3]). Then

$$(\delta^2(\tilde{f}) \equiv 0) \Rightarrow (\exists \tilde{g} \in C^1_{\alpha, \alpha}(\mathcal{W}^q, \mathcal{W}^q); \tilde{f} = \delta^1(\tilde{g})).$$

Using (5.6), (5.7) and Lemma 5.8, we obtain $[\delta, \tilde{f}] = 0$. Then $\tilde{f} \in Z^2(\mathcal{W}^q, \mathcal{W}^q)$.

We have $\tilde{f} = \delta^1(\tilde{g})$, $\hat{f} \equiv 0$, $\bar{f} \equiv 0$, $\bar{v} \equiv 0$ and $\hat{v} \equiv 0$. We deduce, $f = \delta^1(\tilde{g}) + v$. Therefore, $f = \delta^1_{HR}(\tilde{g}) + w$ where $w(\mathcal{W}^q, \mathcal{W}^q) \subset \mathbb{C}$.

So

$$(f \in Z^2(HR, HR)) \Rightarrow (\delta^2_{HR}(w) \equiv 0) \Rightarrow (\delta^2_T(w) \equiv 0).$$

As, $H^2(\mathcal{W}^q, \mathbb{C}) = \mathbb{C}[\varphi_0] \oplus \mathbb{C}[\varphi_1]$ and $\varphi_0 \in B^2_{HR}(\mathcal{W}^q, \mathbb{C})$, we deduce $w \in \mathbb{C}\varphi_1$. ■

5.5 Second cohomology of Special Ramond Hom-superalgebra

Lemma 5.10. *If f is a 2-cocycle, we have*

$$\bar{v} \equiv 0, \quad \hat{f} \equiv 0, \quad \bar{f} \equiv 0, \quad \text{and} \quad \hat{v} \equiv 0.$$

Proof. let $g = \hat{f}(1, \cdot)$.

$$\begin{aligned} [d, f](x_n, x_m, 1) = 0 \Rightarrow \\ -[\alpha(x_n), g(x_m)] + [\alpha(x_m), g(x_n)] + g([x_n, x_m]) + \bar{f}(1, \varphi(x_n, x_m)) \\ - \varphi(\alpha(x_n), g(x_m)) + \varphi(\alpha(x_m), g(x_n)) + \hat{v}(1, [x_n, x_m]) + \bar{v}(1, \varphi(x_n, x_m)). \end{aligned}$$

Since $-\alpha(x_n), g(x_m) + [\alpha(x_m), g(x_n)] + g([x_n, x_m]) + \bar{f}(1, \varphi(x_n, x_m)) \in \mathcal{W}^q$ and $-\varphi(\alpha(x_n), g(x_m)) + \varphi(\alpha(x_m), g(x_n)) + \hat{v}((1, [x_n, x_m]) + \bar{v}(1, \varphi(x_n, x_m))) \in \mathbb{C}$, we deduce

$$-[\alpha(x_n), g(x_m)] + [\alpha(x_m), g(x_n)] + g([x_n, x_m]) + \bar{f}(1, \varphi(x_n, x_m)) = 0. \quad (5.15)$$

$$-\varphi(\alpha(x_n), g(x_m)) + \varphi(\alpha(x_m), g(x_n)) + \hat{v}((1, [x_n, x_m]) + \bar{v}(1, \varphi(x_n, x_m))) = 0. \quad (5.16)$$

In (5.15), the term $\bar{f}(1, \varphi(x_n, x_m))$ is of degree s . The other terms are of degree $n + m + s$. Then if $n + m \neq 0$, we deduce

$$-[\alpha(x_n), g(x_m)] + [\alpha(x_m), g(x_n)] + g([x_n, x_m]) = 0,$$

If $n + m = 0$, we have $\varphi(x_n, x_m) = 0$. Then

$$-[\alpha(x_n), g(x_m)] + [\alpha(x_m), g(x_n)] + g([x_n, x_m]) = 0,$$

which implies that g is an α -derivation. Recall that the set of α -derivations is trivial (see[2]). We deduce $g \equiv 0$ and $\bar{f}(1, \cdot) \equiv 0$.

Since $g \equiv 0$, the equation (5.16) can be write

$$\hat{v}((1, [x_n, x_m]) + \bar{v}(1, \varphi(x_n, x_m))) = 0.$$

If $n + m = -1$ and $s \neq 1$, with $\deg(\hat{v}((1, [x_n, x_m]))) = n + m + s$ and $\hat{v}((1, [x_n, x_m])) \in \mathbb{C}$, we obtain $\hat{v}((1, [x_n, x_m])) = 0$. Thus $\bar{v} \equiv 0$ and $\hat{v} \equiv 0$. ■

Then we obtain the following result about second cohomology of special Ramond Hom-superalgebra.

Theorem 5.11.

$$H^2(SHR, SHR) = \mathbb{C}[\varphi_0].$$

6 Deformations of Virasoro Hom-superalgebras

In this section, we discuss deformations of Ramond Hom-superalgebra and special Ramond Hom-superalgebra.

Definition 6.1. Let $(\mathcal{G}, [\cdot, \cdot]_0, \alpha_0)$ be a Hom-Lie superalgebra. A one-parameter formal deformation of \mathcal{G} is given by the $\mathbb{K}[[t]]$ -bilinear and the $\mathbb{K}[[t]]$ -linear maps $[\cdot, \cdot]_t : \mathcal{G}[[t]] \times \mathcal{G}[[t]] \longrightarrow \mathcal{G}[[t]]$, $\alpha_t : \mathcal{G}[[t]] \longrightarrow \mathcal{G}[[t]]$ of the form

$$[\cdot, \cdot]_t = \sum_{i \geq 0} t^i [\cdot, \cdot]_i \text{ and } \alpha_t = \sum_{i \geq 0} t^i \alpha_i,$$

where each $[\cdot, \cdot]_i$ is an even \mathbb{K} -bilinear map $[\cdot, \cdot]_i : \mathcal{G} \times \mathcal{G} \longrightarrow \mathcal{G}$ (extended to be $\mathbb{K}[[t]]$ -bilinear) and each α_i is an even \mathbb{K} -linear map $\alpha_i : \mathcal{G} \longrightarrow \mathcal{G}$ (extended to be $\mathbb{K}[[t]]$ -linear), and satisfying the following conditions

$$[x, y]_t = -(-1)^{|x||y|} [y, x]_t, \quad (6.1)$$

$$\circlearrowleft_{x,y,z} (-1)^{|x||z|} [\alpha_t(x), [y, z]_t]_t = 0. \quad (6.2)$$

Definition 6.2. Let $(\mathcal{G}, [\cdot, \cdot]_0, \alpha_0)$ be a Hom-Lie superalgebra. Given two deformations $\mathcal{G}_t = (\mathcal{G}, [\cdot, \cdot]_t, \alpha_t)$ and $\mathcal{G}'_t = (\mathcal{G}, [\cdot, \cdot]'_t, \alpha'_t)$ of \mathcal{G} , where

$$[\cdot, \cdot]_t = \sum_{i \geq 0} t^i [\cdot, \cdot]_i, \quad [\cdot, \cdot]'_t = \sum_{i \geq 0} t^i [\cdot, \cdot]'_i, \quad \alpha_t = \sum_{i \geq 0} t^i \alpha_i \text{ and } \alpha'_t = \sum_{i \geq 0} t^i \alpha'_i.$$

We say that they are equivalent if there exists a formal automorphism $\phi_t = \sum_{i \geq 0} t^i \phi_i$,

where $\phi_i \in \left(\text{End}(\mathcal{G}) \right)_0$ and $\phi_0 = \text{id}_{\mathcal{G}}$, such that

$$\phi_t([x, y]_t) = [\phi_t(x), \phi_t(y)]'_t, \quad \forall x, y \in \mathcal{G}, \quad (6.3)$$

$$\text{and } \phi_t \circ \alpha_t = \alpha'_t \circ \phi_t. \quad (6.4)$$

A deformation \mathcal{G}_t is said to be trivial if and only if \mathcal{G}_t is equivalent to \mathcal{G} (viewed as a superalgebra on $\mathcal{G}[[t]]$).

Lemma 6.3. Every deformation HR_t of Ramond Hom-superalgebra such that $[\cdot, \cdot]'_t = [\cdot, \cdot]_0 + \sum_{k \geq p} t^k [\cdot, \cdot]'_k$ and $\alpha_t = \left(\sum_{k \geq 0} a_k t^k \right) \alpha_0$ is equivalent to a deformation

$$[\cdot, \cdot]_t = [\cdot, \cdot]_0 + \left(\sum_{k \geq p} \lambda_k t^k \right) \varphi_1.$$

Proof. Let $HR_t = (HR, [\cdot, \cdot]'_t, \alpha_t)$ be a deformation of Ramond Hom-superalgebra $(HR, [\cdot, \cdot]_0, \alpha_0)$, where

$$[\cdot, \cdot]'_t = [\cdot, \cdot]_0 + \sum_{k \geq p} [\cdot, \cdot]'_k t^k \quad \text{and} \quad \alpha_t = \left(\sum_{k \geq 0} a_k t^k \right) \alpha_0.$$

Condition (6.2) may be written

$$\circlearrowleft_{x,y,z} (-1)^{|x||z|} \sum_{s \geq 0} t^s \left(\sum_{k=0}^s \sum_{i=0}^{s-k} [\alpha_i(x), [y, z]'_k]_{s-i-k}' \right) = 0. \quad (6.5)$$

This equation is equivalent to the following infinite system :

$$\circlearrowleft_{x,y,z} (-1)^{|x||z|} \sum_{k=0}^s \sum_{i=0}^{s-k} [\alpha_i(x), [y, z]_k']_{s-i-k}' = 0, \quad s = 0, 1, \dots \quad (6.6)$$

In particular, for $s = p$,

$$\circlearrowleft_{x,y,z} (-1)^{|x||z|} [\alpha_0(x), [y, z]_0']_p + \circlearrowleft_{x,y,z} (-1)^{|x||z|} [\alpha_0(x), [y, z]_p']_0 = 0.$$

Therefore $[\cdot, \cdot]_p' \in Z^2(HR, HR)$. Since $H^2(HR, HR) = \mathbb{C}[\varphi_1]$, we deduce

$$[\cdot, \cdot]_p' = -\delta^1(\Phi) + \lambda_p \varphi_1, \quad \text{where } \Phi \in C_{\gamma, \gamma}^1, \text{ and } \lambda \in \mathbb{K}.$$

Let $\Phi_t = \Phi_0 + \Phi t^p$, then $\Phi_t^{-1} = \Phi_0 + \sum_{k \geq 1} (-1)^k t^{kp} \Phi^k$.

$$[x, y]_t = \Phi_t^{-1}([\Phi_t(x), \Phi_t(y)]_t').$$

By a simple identification, it follows that $[x, y]_p = \lambda_p \varphi_1$.

As well

$$[\cdot, \cdot]_t = [\cdot, \cdot]_0 + t^p \lambda_p \varphi_1 + \sum_{k > p} t^k [\cdot, \cdot]_k.$$

Thus, by induction we show that $[\cdot, \cdot]_k = \lambda_k \varphi_1 \quad \forall k \geq p$. ■

Theorem 6.4. *Every deformation HR_t such that $[\cdot, \cdot]_t' = [\cdot, \cdot]_0 + \sum_{k \geq p} t^k [\cdot, \cdot]_k'$ and*

$\alpha_t = (\sum_{k \geq 0} a_k t^k) \alpha_0$ *of Ramond Hom-superalgebra is equivalent to a deformation of the form*

$$[\cdot, \cdot]_t = [\cdot, \cdot] + \varphi_0 + t \varphi_1.$$

Proof. It follows from

$$[L_n, G_m]_t = [L_n, G_m] + \sum_{i \geq p} a_i \varphi_1(L_n, G_m) t^i;$$

$$[L_n, L_m]_t = [L_n, L_m] + \varphi_0(L_n, L_m);$$

$$[L_n, G_m]_t' = [L_n, G_m] + \lambda \varphi_1(L_n, G_m) t;$$

$$[L_n, L_m]_t' = [L_n, L_m] + \varphi_0(L_n, L_m).$$

$\Phi_0 = id$; $\Phi_s(1) = 0, \forall s > 0$; $\Phi_s(G_m) = \frac{a_{s+1}}{\lambda} G_m, \forall s \geq p$; $\Phi_s(L_n) = 0, \forall s > 0$. Thus $[\cdot, \cdot]_t = [\cdot, \cdot] + \varphi_0 + \lambda t \varphi_1$. By rescaling we obtain the desired result. ■

Theorem 6.5. *Every deformation SHR_t such that $[\cdot, \cdot]_t' = [\cdot, \cdot]_0 + \sum_{k \geq p} t^k [\cdot, \cdot]_k'$ and*

$\alpha_t = (\sum_{k \geq 0} a_k t^k) \alpha_0$ *of special Ramond Hom-superalgebra is equivalent to a deformation*

$$[\cdot, \cdot]_t = [\cdot, \cdot] + \varphi_1 + t \varphi_0.$$

Remark 6.6. Theorem 6.5 can be proved in the same way as Theorem 6.4.

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